CONCERNING SIMPLE CONTINUOUS CURVES*

BY

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1. Introduction

Various definitions of simple continuous arcs and closed curves have been given.† The definitions of arcs usually contain the requirement that the point-set in question should be bounded. In attempting to prove that every interval t of an open curve as defined in a recent paper‡ is a simple continuous arc, while I found it easy to prove that t satisfies all the other requirements of Janiszewski's definition (modified as indicated below) it was only by a rather lengthy and complicated argument that I succeeded in proving that it satisfies the requirement of boundedness. In Lennes' definition the requirement of boundedness is superfluous.§ However I found it difficult to prove that t satisfies a certain one of the other requirements of this definition, namely that the point-set in question should contain no proper connected subset that contains both A and B. In the present paper I will give a definition|| of a simple continuous arc which stipulates neither that the set M should be bounded nor that it should contain no proper connected subset containing both A and B. I will show that, in a euclidean space of two dimensions, every point-set that satisfies this definition is an arc in the sense of Jordan. It is easy to prove¶ that every interval of an open curve satisfies this definition.

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† Cf., for example, the following: S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, 2e série, vol. 16 (1911–12), pp. 79–170. N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), p. 308 and Bulletin of the American Mathematical Society, vol. 12 (1906), p. 284. W. Sierpinski, *L'arc simple comme un ensemble de points dans l'espace á m dimensions*, Annali di Matematica, Serie III, vol. 26 (1916), pp. 131–150. J. R. Kline, *Concerning the relation between approachability and connectivity in kleinem*. R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370.

‡ R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 159.

§ Cf. G. H. Hallett, Jr., *Concerning the definition of a simple continuous arc*, Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 325–326.

|| Definition 1 below.

¶ See proof of Theorem 3 below.

In connection with certain problems where the boundedness of the point-set in question is not presupposed, but where relatively more information is at hand concerning *connectedness,* it seems likely that Definition 1 may be more useful than that of Janiszewski. On the other hand Janiszewski's definition (or a modification of it*) may be of more use in certain cases where one is concerned with sets that are known in advance to be bounded (or can easily be proved to be bounded) but concerning which less is known in advance with regard to connectedness.†

In the latter part of § 2, a very simple characterization of a simple closed curve is given. It is defined merely as a closed connected and bounded point-set which is disconnected by the omission of any two of its points.‡

In § 3 the problem of defining simple continuous arcs, closed and open curves and rays is approached from a different point of view. With the use of the notion of the boundary of a point-set M with respect to a point-set that contains M, conditions are given which a point-set must satisfy in order that it should be a *simple continuous curve* (that is to say one of the four types of curves mentioned above). This classification of the general notion *simple continuous curve* having been given, it is easy to so particularize it as to obtain a characterization of any given one of the four special types of simple continuous curves. A definition of an open curve from this point of view can be obtained from that of a closed curve by the mere substitution of the word "bounded" in place of the word "proper."

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DEFINITION 1. If A and B are two distinct points, a simple continuous arc from A to B is a closed, connected set of points M containing A and B such that (1) $M - A$ and $M - B$ are connected, (2) if P is any point of M

* Cf. Definition 2 below.

† In this connection see the article by Sierpinski referred to above. His definition does not require explicitly even that M itself should be connected. It requires boundedness however and also certain positive information concerning the relation of the endpoints A and B to the set M. In my Definition 2 the phrase " except the points A and B " is to be interpreted not as meaning that the points A and B do not fulfill the requirements indicated but merely as leaving the question open whether they do or do not fulfill these requirements. Sierpinski definitely stipulates that M is not the sum of two closed point-sets having only A, or only B, in common and each consisting of more than one point. If this stipulation were omitted his definition would not completely characterize a simple continuous arc and would indeed apply to some sets that are not connected, e.g., to a set composed of three distinct points.

‡ Lennes defines a simple closed curve as " the set of points consisting of two continuous arcs, each connecting a pair of distinct points A and B and having no other point in common." This definition presupposes a previous definition of a simple continuous arc. Janiszewski (loc. cit., p. 137) defines " *Une ligne simple fermée* Γ " as " un continu qui peut être décomposé en deux continus e_1 et e_2 n'ayant en commun que deux points M et N arbitrairement donnés sur Γ."

distinct from A and from B then $M - P$ is the sum of two mutually exclusive connected point-sets neither of which contains a limit point of the other one.

If for a point O of a connected point-set M the set $M - O$ is the sum of two mutually exclusive point-sets M_1 and M_2 neither of which contains a limit point of the other one then M_1 and M_2 will be called sects* (of M) from O, and M will be said to be disconnected by the omission of O and will be said to be separated by the omission of O into the two sets M_1 and M_2. If P is a point of M distinct from O then in case there is only one sect of M from O that contains P that sect will be called the sect OP. If at the same time there is only one sect from P that contains O the set of all those points of M that are common to the sects OP and PO will be called the *segment* OP (of M) while the set of points consisting of all the points of the segment OP together with its *endpoints* (O and P) will be called the *interval* OP (of M).

THEOREM 1. *In a Euclidean space of two dimensions every set of points M that satisfies the requirements of Definition* 1 *is an arc in the sense of Jordan.*

Proof. Suppose M is a set of points satisfying all the requirements of Definition 1. If O is any point of M distinct from A and from B it is easy to see that there are only two sects of M from O. One of these sects contains A and the other one contains B. For suppose that one of them contains both A and B. Let OC denote the other one. Let K denote the set of all points $[X]$ such that one sect from X contains neither A nor B but has at least one point in common with OC. For every point P of K let K_P denote that sect from P which contains neither A nor B and let $\overline{K}_P$ denote the other sect from P.

The points of OC can† be arranged in a well-ordered sequence β. Let P_1 be the first point in the sequence β. Let P_2 be the first point of β which lies in K_{P_1}. Let P_3 be the first point of β which is common to K_{P_1} and K_{P_2}. Let P_4 be the first point of β which is common to K_{P_1}, K_{P_2}, and K_{P_3}. This process may be continued. It follows that the sequence β contains a well-ordered subsequence α such that if T is a subset of the elements of α there is an element of α which follows, in α, all the elements of T if, and only if, there exists a point which belongs to K_P for every point P of T and, if there does exist such a point, then the first element of α that follows all the elements of T is the first element of β which belongs to K_P for every point P of T. Suppose that X and Y are distinct points of α and that Y precedes X in α. Then X is in K_Y and therefore is not in $\overline{K}_Y + Y$ and hence $\overline{K}_Y + Y$ must

* The term "sect" is used by Halsted with a somewhat different meaning. Cf. G. B. Halsted, *Rational Geometry*, Wiley and Sons, New York, 1904.

† The Zermelo Postulate is here assumed. Cf. E. Zermelo, *Beweiss, dass jede Menge wohlgeordnet sein kann*, Mathematische Annalen, vol. 59 (1904), pp. 514–516. Concerning this postulate cf. Philip E. B. Jourdain, Comptes Rendus, vol. 166 (1918), pp. 520–523 and 984–986.

lie wholly in K_X or wholly in $\bar{K}_X$ (otherwise $\bar{K}_Y + Y$ would not be connected and consequently $K_Y + Y + \bar{K}_Y$ would not be connected). But $\bar{K}_Y$ contains A and K_X does not. Hence $\bar{K}_Y + Y$ is a subset of $\bar{K}_X$. Therefore K_X contains no point of $\bar{K}_Y + Y$. It follows that K_X is a subset of K_Y. But of any two points in α one of them precedes the other one in α. Hence if X and Y are two points of α either K_X contains K_Y or K_Y contains K_X.

If X and Y are distinct points of K and K_X contains K_Y then K_Y does not contain K_X. For since $X + K_X$ is closed, $X \neq Y$ and Y is a limit point of K_Y, therefore Y is in K_X. But Y is not in K_Y. Hence K_Y does not contain K_X.

If, for two distinct points X and Y of the set K, the sect K_X contains the sect K_Y then X will be said to precede Y in K. In view of the results established above it is clear that if X and Y are two distinct points in K then (1) if X precedes Y, Y does not precede X, (2) if X precedes Y and Y precedes Z then X precedes Z, (3) if X and Y are both elements of α then either X precedes Y or Y precedes X.

The ray OC is unbounded. For suppose it is bounded. Then the family of all sets $K_X + X$ for all points X of α is a family of closed, bounded point-sets such that of every two of them one contains the other one. It follows by a theorem established in a recent paper* that there exists at least one point W which belongs to $K_X + X$ for every point X of α. Let W_1 denote the first such point W in the sequence β. Then W_1 is an element of α that follows all the elements of α. Thus the supposition that OC is bounded leads to a contradiction.

Suppose that X and Y are two distinct points of K that do not belong to $O + OC$. Since the connected set $O + OC$ contains a point of K_X but does not contain X therefore $O + OC$ is a subset of K_X. Similarly $O + OC$ is a subset of K_Y. Thus K_X and K_Y have at least one point in common. Suppose now that K_X is not a subset of K_Y. Then since $X + K_X$ is connected and contains at least one point of K_Y, K_X must contain Y. Hence $X + \bar{K}_X$ does not contain Y. But $\bar{K}_X + X$ is connected and contains the point A in common with $\bar{K}_Y$. Hence $\bar{K}_X + X$ is a subset of $\bar{K}_Y$ and therefore $K_Y + Y$ is a subset of K_X. Thus it is proved that if X and Y are two distinct points of $K - OC - O$ then either X precedes Y or Y precedes X.

Let H denote the set of points composed of K together with the set of all points $[Y]$ such that Y belongs to K_X for some point X of K.

Since M is connected either H contains a limit point of $M - H$ or $M - H$ contains a limit point of H. Suppose that $M - H$ contains a point Z which

* R. L. Moore, *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true*, Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 206–210. Cf. also S. Janiszewski, loc. cit.

is a limit point of H. Then Z is the sequential limit point of some infinite sequence of distinct points $Z_1, Z_2, Z_3, \cdots$ belonging to H. It follows that there exists an infinite sequence of distinct points $X_1, X_2, X_3, \cdots$ belonging to $K - O - OC$ and an infinite sequence of distinct positive integers $n_1, n_2, n_3, \cdots$ such that, for every m, (1) Z_{n_m} belongs to K_{X_m} and (2) X_{m+1} precedes X_m in K. Let K^* denote the point-set $X_1 + X_2 + X_3 + \cdots$. It is clear that (1) if Y is a point of H there exists a positive integer m such that K_{X_m} contains Y, (2) $M - H$ contains every limit point of K^*. Every limit point of H which lies in $M - H$ is a limit point of K^*. Suppose $M - H$ contains two points D and E which are limit points of H. Then it can be shown that there exist in K^* two sequences of points $D_1, D_2, D_3, \cdots$ and $E_1, E_2, E_3, \cdots$ such that (1) D and E are sequential limit points of $D_1, D_2, D_3, \cdots$ and of $E_1, E_2, E_3, \cdots$ respectively, (2) for every i, E_i is preceded by D_i and D_i is preceded by E_{i+1} in K. There exist five regions§ R_1, R_2, R_3, R_4, and R_5 such that (1) R_3 contains D and R_4 contains E, (2) R_3' is a subset of R_2, R_4' is a subset of R_1 and R_1' is a subset of R_5, (3) R_5' and R_2' have no point in common. There exists a positive integer n such that for every positive integer m the point D_{n+m} is in R_3 and the point E_{n+m} is in R_4. It is clear that the intervals $E_{n+1} D_{n+1}, E_{n+2} D_{n+2}, E_{n+3} D_{n+3}, \cdots$ of M are all closed connected subsets of K and that no two of them have a point in common. For every m the interval $E_{n+m} D_{n+m}$ contains a closed connected subset t_m that contains at least one point on the boundary of R_1 and at least one point on the boundary of R_4 but contains no point without R_1 or within R_4. No two point-sets belonging to the infinite sequence $t_1, t_2, t_3, \cdots$ have a point in common.† It follows that there exists (1) an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ such that, for every j, $n_{j+1} > n_j$ and (2) a closed connected set of points t and a sequence of closed connected point-sets $k_{n_1}, k_{n_2}, k_{n_3}, \cdots$ such that, for every j, k_{n_j} is a subset of t_{n_j} and such that (a) each of the point-

§ In my paper *On the foundations of plane analysis situs*, loc. cit. (this paper will be referred to as F.A.), the notion *point* is undefined and *region* is also undefined except in so far as it is understood that every region is some sort of collection of points. A considerable part of the present argument holds good not only for a euclidean space of two dimensions but for any space satisfying Axioms 1′ and 4 of F.A. The whole of this argument holds good for every space satisfying the set of Axioms Σ_1 of F.A. Every such space is, however, in one to one continuous correspondence with an ordinary euclidean space of two dimensions; cf. my paper *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), pp. 169–178. If one desires to read the present paper without special reference to any particular system of axioms he may think of the word *region* as applying to any bounded connected domain in a euclidean space of two dimensions.

† Up to this point the present proof holds good for every space satisfying Axioms 1′ and 4 of F.A. It therefore holds good for all euclidean spaces (of however many dimensions) as well as for many other spaces including certain spaces that are neither metrical, descriptive, nor separable (cf. F.A., p. 131). If the statement in the next sentence (which can be easily established for euclidean space of two dimensions) can be proved to hold true in euclidean space of any number of dimensions then the present proof holds good for any such space.

sets $t, k_{n_1}, k_{n_2}, k_{n_3}, \cdots$ is a subset of $R_1' - R_4$ but contains at least one point on the boundary of R_1 and at least one point on the boundary of R_4, (b) if $P_{n_1}, P_{n_2}, \cdots$ is a sequence of points such that, for every j, P_{n_j} belongs to k_{n_j} then t contains every limit point of the point-set $P_{n_1} + P_{n_2} + P_{n_3} + \cdots$, (c) if $n_1, \bar{n}_2, \bar{n}_3, \cdots$ is an infinite sequence of distinct integers belonging to the set $n_1, n_2, n_3, \cdots$ and P is a point of t there exists an infinite sequence of points $P_{\bar{n}_1}, P_{\bar{n}_2}, P_{\bar{n}_3}, \cdots$ such that, for every j, $P_{\bar{n}_j}$ belongs to $k_{\bar{n}_j}$ and such that P is the sequential limit point of the sequence $P_{\bar{n}_1}, P_{\bar{n}_2}, P_{\bar{n}_3}, \cdots$. If T is a point of t, T is the sequential limit point of some sequence of points $T_{n_1}, T_{n_2}, T_{n_3}, \cdots$ such that, for every j, T_{n_j} belongs to k_{n_j}. Of the two sects of M from T one must contain an infinite subsequence $T_{n_{j_1}}, T_{n_{j_2}}, T_{n_{j_3}}, \cdots$ of the sequence $T_{n_1}, T_{n_2}, T_{n_3}, \cdots$. For every positive integer m the connected point-set $k_{n_{j_m}}$ contains $T_{n_{j_m}}$ but does not contain T. It follows that that sect from T which contains the points $T_{n_{j_1}}, T_{n_{j_2}}, T_{n_{j_3}}, \cdots$ contains also the point-sets $k_{n_{j_1}}, k_{n_{j_2}}, k_{n_{j_3}}, \cdots$. But every point of t is a limit point of $k_{n_{j_1}} + k_{n_{j_2}} + \cdots$. Hence that sect contains every point of $t - T$. For every point T of t let $\bar{R}_T$ denote that sect from T which contains $t - T$ and let R_T denote the other sect from T. If T_1 and T_2 are two distinct points of t, $\bar{R}_{T_1}$ contains the point T_2 of the connected point-set $R_{T_2} + T_2$ and T_1 does not belong to $R_{T_2} + T_2$. It follows that R_{T_2} is a subset of $\bar{R}_{T_1}$. Hence R_{T_1} and R_{T_2} have no point in common. Hence there do not exist more than two points X belonging to t such that R_X contains A or B. Let t_0 denote $t, t - X_1$, or $t - (X_1 + X_2)$ according as there are no such points X or there is only one such point X_1 or there are two such points X_1 and X_2. For each point T of t_0 the sect R_T is unbounded (cf. above) and contains a point (T) in R_1'. Hence it contains at least one point P_T on the boundary of R_5. Consider the set L of all P_T's for all points T of t_0. There is a one to one correspondence between the point-sets L and t_0. It follows that L is uncountable. But no point of any sect R_T is a limit point of $M - R_T$. Hence no point of L is a limit point of L. But every uncountable set of points α contains at least one point which is a limit point of α. Thus the supposition that $M - H$ contains more than one limit point of H has led to a contradiction.

It is clear that H cannot contain more than one limit point of $M - H$. For no point of OC is a limit point of $M - OC$ and if X and Y are two points of H not belonging to OC there exist two points $\bar{X}$ and $\bar{Y}$ belonging to K such that $\bar{X} + K_{\bar{X}}$ contains X and $K_{\bar{Y}} + \bar{Y}$ contains Y. But either $K_{\bar{X}}$ contains $K_{\bar{Y}} + \bar{Y}$ or $K_{\bar{Y}}$ contains $\bar{K}_{\bar{X}} + \bar{X}$. In the first case Y is not a limit point of $\bar{K}_{\bar{X}}$ and therefore is not a limit point of $M - H$ which is a subset of $\bar{K}_X$. In the second case X is not a limit point of $M - H$.

It follows that there exists one and only one point O which belongs to one of the sets H and $M - H$ and is a limit point of the other one.

Suppose that O belongs to $M - H$. Neither of the sets H and $M - (H + O)$ contains a limit point of the other one. But K is connected and each sect from O is connected. It follows that K is one of the sects of M from O. Thus the supposition that O belongs to $M - H$ leads to a contradiction. Hence O belongs to H. Thus for every point P of M such that one sect from P contains neither A nor B there exists a point O_P such that (1) the sect $O_P P$ contains neither A nor B, (2) the sect $O_P P$ is not a subset of any other sect that contains neither A nor B. Let $\overline{H}$ denote the set of all points P of the segment AB such that one sect from P contains neither A nor B. Let N_1 denote the set of all points O_P for all points P of $\overline{H}$. For each point X of $\overline{H}$ let M_X denote that sect from X which contains neither A nor B. Let N_2 denote the set of all those points of M that belong neither to N_1 nor to any M_X for any point X of N_1. Let N denote the set $N_1 + N_2$. Every point of N_1 will be called an improper point and every point of N_2 will be called a proper point. Either N_2 contains a limit point of $\overline{H}$ or $\overline{H}$ contains a limit point of N_2. Suppose first that N_2 contains a point O which is a limit point of $\overline{H}$. Then there exists a sequence of distinct points $X_1, X_2, X_3, \cdots$ belonging to N_1 and a sequence of points $P_1, P_2, P_3, \cdots$ such that O is the sequential limit point of the sequence $P_1, P_2, P_3, \cdots$ and such that, for every n, P_n either coincides with X_n or belongs to M_{X_n}. There exist about O two regions $\overline{R}_1$ and $\overline{R}_2$ such that $\overline{R}_2'$ is a subset of $\overline{R}_1$. There exists a positive integer n such that the points $P_n, P_{n+1}, P_{n+2}, \cdots$ are all in $\overline{R}_2$. But each of the point-sets $P_n + M_{P_n}$, $P_{n+1} + M_{P_{n+1}}$, $P_{n+2} + M_{P_{n+2}}$, $\cdots$ is closed, connected, and unbounded. It follows that for each m, $M_{P_{n+m}}$ contains a closed, connected subset $\bar{t}_m$ which contains at least one point on the boundary of $\overline{R}_1$ and at least one point on the boundary of $\overline{R}_2$ and every point of which is either on the boundary of $\overline{R}_1$ or of $\overline{R}_2$ or in the domain $\overline{R}_1 - \overline{R}_2'$. No two of the point-sets $\bar{t}_1, \bar{t}_2, \bar{t}_3, \cdots$ have a point in common. That this leads to a contradiction follows by an argument analogous to (or identical with) that used in a similar connection above.

Suppose secondly that $\overline{H}$ contains a point O which is a limit point of N_2. Clearly O must belong to N_1. If X is a point of N_2 distinct from A and from B and Y is a point belonging either to N_1 or to N_2, X will be said to precede Y or to follow Y according as Y belongs to the sect XB or to the sect XA. It may be easily proved that there exists in N_2 a sequence of distinct points $X_1, X_2, X_3, \cdots$, all distinct from A and from B, such that O is a sequential limit point of this sequence and such that either (1) for every n, X_n precedes O and X_{n+1} or (2) for every n, X_n follows O and X_{n+1}. Suppose that, for every n, X_n precedes O and X_{n+1}. Each of the intervals $X_1 X_2$, $X_2 X_3$, $\cdots$ is closed and connected, no two of them contain in common any point other than a common endpoint and no one of them contains the point O. That O is

the only limit point of $X_1 X_2 + X_2 X_3 + \cdots$ that does not belong to $X_1 X_2 + X_2 X_3 + \cdots$ may be proved by an argument similar to that used above in the proof that $M - H$ does not contain more than one limit point of H.

Now sect

$$OA = M - M_O - O = (\overline{AX_1}^* + \overline{X_1 X_2} + \overline{X_2 X_3} + \cdots) + (B + V)$$

where V is the point-set $M - (B + M_O + O + \overline{AX_1} + \overline{X_1 X_2} + \overline{X_2 X_3} + \overline{X_3 X_4} + \cdots)$. Neither of the two sets $(\overline{AX_1} + \overline{X_1 X_2} + \overline{X_2 X_3} + \cdots)$ and $(B + V)$ contains a limit point of the other one. Hence the sect OA is not connected. This involves a contradiction. Hence it is not true that, for every n, X_n precedes O and X_{n+1}. In an entirely similar way it may be proved that X_n cannot follow O and X_{n+1} for every n. Thus the supposition that $\overline{H}$ contains a limit point of N_2 has led to a contradiction.

It follows that the set $\overline{H}$ does not exist. Hence if P is any point of $M - (A + B)$, $M - P$ is the sum of two sects of which one contains A and the other contains B. It follows that the two sects PA and PB have no point in common and neither of them contains a limit point of the other one. Suppose now that $\overline{M}$ is a proper subset of M that contains both A and B. Then M contains a point P that does not belong to $\overline{M}$. Let $\overline{M}_1$ denote the set of all points common to $\overline{M}$ and the sect PA and let $\overline{M}_2$ denote the set of all points common to $\overline{M}$ and the sect PB. Neither of the sets $\overline{M}_1$ and $\overline{M}_2$ contains a limit point of the other one. But $\overline{M} = \overline{M}_1 + \overline{M}_2$. Hence $\overline{M}$ is not connected. Thus M contains no proper connected subset that contains both A and B. It follows that M satisfies all the requirements of Lennes' definition of an arc with the exception of the requirement that it should be bounded. That it also satisfies the latter requirement follows from the theorem of Hallett referred to above. The truth of Theorem 1 is therefore established.

DEFINITION 2.† If A and B are two distinct points a *simple continuous arc* from A to B is a closed, connected, and bounded point-set containing A and B which is disconnected by the omission of any one of its points which is distinct from A and from B.

THEOREM 2. *In a space satisfying Axioms‡ 1′ and 4 of F.A. every point-*

* The notation $\overline{AB}$ will be used to denote the interval AB of M.

† This definition is closely related to those of Janiszewski and Sierpinski, loc. cit. Janiszewski's definition contains an unnecessary requirement concerning connectedness and also the requirement that the point-set M should be an "irreducible continu from A to B" i.e., that it should contain no proper closed and connected subset containing both A and B. He indicates that this latter condition is redundant and makes reference in this connection to a proof of another theorem. I have not succeeded however in seeing that the argument given there proves the redundancy in question. As I have already observed, Sierpinski's definition contains a certain stipulation concerning A and B.

‡ Loc. cit., pp. 163 and 132.

set M that satisfies the requirements of Definition 2 satisfies also those of Lennes' definition.

Proof. Let M be a set of points satisfying the requirements of Definition 2. Certain portions of the proof of Theorem 1 clearly apply here. In particular the same argument that was used there applies here to show that if P is any point of M distinct from A and from B and $M - P$ is the sum of two subsets neither of which contains a limit point of the other one then if one of these subsets contains neither A nor B it must be unbounded. But here M itself is bounded. Therefore, for every point P of $M - (A + B)$, $M - P$ is* the sum of two mutually separated† point-sets M_A and M_B such that M_A contains A and M_B contains B. It follows‡ that M contains no proper connected subset containing both A and B. The truth of Theorem 2 is therefore established.

DEFINITION 3. An *open curve* is a closed and connected point-set which is separated into two connected subsets by the omission of any one of its points.

If P is a point of an open curve M the point-set obtained by adding P to either of the two sects into which M is separated by the omission of P is called a *ray*. The two rays of M so determined by a point P are said to start from P. If A is a point of M distinct from P that ray of M which starts from P and contains A will be called the ray PA.

THEOREM 3. *In euclidean space of two dimensions, if A and B are two points of the open curve M the interval AB of M is a simple continuous arc from A to B.*§

Proof. It is clear that the interval $\overline{AB}$ is closed. It is connected. For suppose it is not. Then it is the sum of two mutually separated point-sets. Let M_A denote that one of these sets which contains A and let $\overline{M}_A$ denote the other one. Let AC denote that ray from A which does not contain B and let BD denote that ray from B which does not contain A. If $\overline{M}_A$ contains B then neither of the complementary sets $AC + M_A$ and $\overline{M}_A + BD$ contains a limit point of the other one. If $\overline{M}_A$ does not contain B then neither of the complementary sets $AC + BD + M_A$ and $\overline{M}_A$ contains a limit point of the other one. Thus in either case M is not connected. Thus the supposition that $\overline{AB}$ is not connected has led to a contradiction.

* Cf. an argument given by Sierpinski, loc. cit. pp. 137–140. His argument assumes separability while the proof given here holds good in every space satisfying Axioms 1′ and 4. Such spaces are of course not necessarily separable.

† Two point-sets are said to be *mutually separated* if neither contains a point or a limit point of the other one.

‡ See latter part of proof of Theorem 1.

§ The definition of an open curve given on page 159 of F.A. is (aside from phraseology) equivalent to Definition 3. Theorem 49 is however not fully proved in F.A. In particular it is there assumed without proof that t contains no proper connected subset that contains both A and B.

Suppose that O is any point of the interval $\overline{AB}$ distinct from A and from B. It will be shown that $\overline{AB} - O$ is the sum of two connected point-sets neither of which contains a limit point of the other one. Suppose that one of the sects of M from O contains both A and B. Let OX denote the other sect of M from O. Let N denote the set of all those points which are common to the sect OA and the interval $\overline{AB}$. The point-set N contains A and B. Suppose it is not connected. Then it is the sum of two sets neither of which contains a limit point of the other one. Let N_A denote that one of these sets which contains A and let N_B denote the other one. If N_A contains B then the sect OA is the sum of the two mutually separated sets $AC + BD + N_A$ and N_B. If N_A does not contain B then the sect OA is the sum of the two mutually separated sets $AC + N_A$ and $BD + N_B$. In either case the sect OA is not connected. Thus the supposition that one sect of M from O contains both A and B leads to a contradiction. It follows that the sects OA and OB have no point in common. Hence neither of the point-sets $\overline{OA} - O$ and $\overline{OB} - O$ contains a point or a limit point of the other one. It is easy to see that these point-sets are connected and that their sum is $\overline{AB} - O$. Thus if O is any point of the interval AB distinct from A and from B then $\overline{AB} - O$ is the sum of two connected point-sets neither of which contains a point or a limit point of the other one. Thus the interval $\overline{AB}$ satisfies all the requirements for an arc as given in Definition 1.

For further results concerning open curves see F.A., loc. cit., and a paper by J. R. Kline.*

DEFINITION 4. A simple closed curve is a closed, connected, and bounded point-set which is disconnected by the omission of any two of its points.

THEOREM 4. *In a space satisfying Axioms* 1′ *and* 4 *of F.A., a set of points M satisfying the requirements of Definition* 4 *is the sum of two simple continuous arcs that have only their endpoints in common.*

Proof. Suppose the set of points M satisfies the requirements of Definition 4. Let A and B denote two distinct points of M. By hypothesis the set $M - (A + B)$ is the sum of two mutually separated point-sets M_1 and M_2. I will show that $M_1 + A + B$ and $M_2 + A + B$ are simple continuous arcs from A to B. That $M_1 + A + B$ is closed is evident.

It is also connected. For suppose it is not. Then it is the sum of two closed, mutually exclusive point-sets N and K. If N should contain both A and B then M would be the sum of the two separated sets $N + M_2$ and K which is contrary to hypothesis. Similarly K cannot contain both A and B. It follows that one of the sets N and K contains A and the other contains B. Suppose K contains A. The set $M_2 + A + B$ must be connected. For if

* J. R. Kline, *The converse of the theorem concerning the division of a plane by an open curve*, these Transactions, vol. 18 (1917), pp. 177–184.

it were the sum of two mutually separated sets $\overline{N}$ and $\overline{K}$ where $\overline{K}$ contains both A and B then M would be the sum of the two mutually separated sets $\overline{K} + M_1$ and $\overline{N}$, while if it were the sum of two mutually separated sets $\overline{N}$ and $\overline{K}$ where $\overline{N}$ contains B and $\overline{K}$ contains A then M would be the sum of the two mutually separated sets $K + \overline{K}$ and $N + \overline{N}$. The set K is connected. For otherwise it would be the sum of two mutually separated sets K_1 and K_2 (where K_1 contains A) and M would be the sum of two separated sets $M_2 + K_1 + N$ and K_2. Likewise N is connected. Hence $N + M_2 + A$ is connected. Either K or N contains more than one point.

Case I. Suppose that K contains more than one point but N contains only the point B and that the set $M_2 + A$ is not connected. Then $M_2 + A$ is the sum of two mutually separated point-sets L_1 and L_2 where L_2 contains A, B is a limit point both of L_1 and of L_2, the set $M - B$ is the sum of the two mutually separated sets L_1 and $L_2 + K$, and the set $M - (A + B)$ is the sum of the two separated sets $L_2 - A$ and $K - A + L_1$. That this leads to a contradiction may be proved by an argument entirely analogous to that employed in the sub-case of Case II below in which N contains more than one point.*

Case II. Suppose that K contains more than one point and that either $M_2 + A$ is connected or N contains more than one point. In the first case let Y_0 denote the point B. Otherwise let Y_0 denote some definite point of N other than B. In either case, if X is any point of K other than A, $M - X - Y_0$ is the sum of two mutually separated sets M_{XY_0} and $\overline{M}_{XY_0}$ where M_{XY_0} contains $M_2 + A$. The set $\overline{M}_{XY_0} + X + Y_0$ is the sum of two sets K_X and N_{Y_0} where K_X is a subset of K and N_{Y_0} is a subset of N. For every point X of K distinct from A, the set K_X is connected. For otherwise it would be the sum of two separated sets K_{X1} and K_{X2} where K_{X1} contains X and in this case the set M would be the sum of two mutually separated sets, K_{X2} and $N + M_{XY_0} + K_{X1}$, which is contrary to hypothesis. The set $M_{XY_0} + X + Y_0 + N_{Y_0}$ is connected. For if it were the sum of two mutually separated sets L and T where L contains X then M would be the sum of two mutually separated sets, $L + K_X$ and T, which is contrary to hypothesis. Suppose that X_1 and X_2 are two points of K and that X_2 is in K_{X_1}. If X_1 were in K_{X_2} then the connected point-set $M_{X_2Y_0} + X_2 + Y_0 + N_{Y_0}$ would contain one point X_2 of K_{X_1} but would not contain the point X_1 and therefore would necessarily be a subset of K_{X_1} and therefore of K which is not the case. Hence X_1 is not in K_{X_2}. But K_{X_2} is connected and, by hypothesis, contains a point X_2 in K_{X_1}. Hence K_{X_2} is a subset of K_{X_1}. Suppose now that X_1 and X_2 are two points in K and that K_{X_1} and K_{X_2} have a point in common. Then, unless X_2 is in K_{X_1}, K_{X_2} must contain the whole of K_{X_1}. But if X_2 is in

* In the proof in question below merely replace N by L_1 and M_2 by $L_2 - A$.

K_{X_1} then, as has been shown above, K_{X_1} contains the whole of K_{X_2}. We thus have the result that if K_{X_1} and K_{X_2} have a point in common then one of them contains the other one. The points of K can be arranged in a well-ordered sequence β. Let P_1 be the first point in the sequence β. Let P_1 be the first point of β which lies in K_{P_1}. Let P_3 be the first point of β which is common to K_{P_1} and K_{P_2}. Let P_3 be the first point of β which is common to K_{P_1}, K_{P_2}, and K_{P_3}. This process may be continued. It follows that the sequence β contains a subsequence α such that if T is a subset of the elements of α there is an element of α which follows, in α, all the elements of T if, and only if, there exists a point which belongs to K_P for every point P of T and, if there does exist such a point, then the first element of α that follows all the elements of T is the first point of β which belongs to K_P for every point P of T. For every two distinct points X and Y of the sequence α either K_X contains K_Y or K_Y contains K_X. Moreover the set K is bounded. It follows* that there exists a point P which belongs to every K_X for every point X of α. It is clear that there is only one such point P and that K_P consists of the single point P. For each point X of α the set $M_{XY_0} + X + Y_0$ is connected. But every point of $M - P$ belongs to $M_{XY_0} + X + Y_0 + N_{Y_0}$ for some point X of α. It follows that both $M_{PY_0} + Y_0$ and $M - P$ are connected. If Y is any point of N_{Y_0} then $\overline{M}_{PY} + P + Y = P + N_Y$. By an argument similar to that used to establish the existence of P it can be shown that there exists in N a point Q such that $N_Q = Q$. The set $M - (P + Q)$ is connected. Thus the supposition that $M_1 + A + B$ is not connected has led to a contradiction. Similarly $M_2 + A + B$ is connected.

Suppose now that P_1 is any point of M_1. Let P_2 denote any point of M_2. The set $M - (P_1 + P_2)$ is the sum of two mutually separated sets $\overline{M}_1$ and $\overline{M}_2$. Since $\overline{M}_1 + P_1 + P_2$ and $\overline{M}_2 + P_1 + P_2$ are connected, P_1 is a limit point both of $\overline{M}_1$ and of $\overline{M}_2$. It follows that both of these sets contain points of M_1. Let $\overline{N}_1$ denote the set of points common to $\overline{M}_1$ and $M_1 + A + B$ and let $\overline{N}_2$ denote the set of points common to $\overline{M}_2$ and $M_1 + A + B$. The set $(M_1 + A + B) - P_1$ is the sum of the two mutually separated sets $\overline{N}_1$ and $\overline{N}_2$.

It has now been shown that the point-set $M_1 + A + B$ satisfies all the requirements of a simple continuous arc from A to B as given in Definition 2. Similarly $M_2 + A + B$ is a simple continuous arc from A to B. Clearly these two arcs have only A and B in common. The truth of Theorem 4 is therefore established.

* See my paper, *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true*, loc. cit.

3

DEFINITION. If the point-set M is a proper subset of the point-set N, the *boundary* of M with respect to N is the set of all points $[X]$ such that X is either a point or a limit point of M and also either a point or a limit point of $N - M$.

THEOREM 5. *If the continuous* point-set M contains no continuous set of condensation† then every two points of M are the extremities of a simple continuous arc that lies wholly in M.*

Indication of proof. By an argument largely similar to (but not entirely identical with) one used in my paper, *Concerning continuous sets that have no continuous sets of condensation*,‡ it may be proved that M is "connected in kleinem." By an argument similar to that used in my paper, *A theorem concerning continuous curves*,§ it may be proved that every two points of M are the extremities of a simple continuous arc lying wholly in M.

THEOREM 6. *In euclidean space of two dimensions if no continuous subset of the continuous point-set M has more than two boundary points with respect to M then M is a simple continuous arc, a simple closed curve, a simple open curve, or a ray of a simple open curve.*

Proof. The set M contains no continuous set of condensation. Hence by Theorem 5 every two points of M are the extremities of at least one simple continuous arc that lies wholly in M. Let G denote the family of all arcs which lie in M. If AB is an arc of G no point of AB distinct from A and from B is a limit point of $M - AB$. For if there should exist such a point P there would exist on AB two points $\bar{A}$ and $\bar{B}$ in the order $A\bar{A}P\bar{B}B$ on AB and the interval $\bar{A}\bar{B}$ would be a continuous subset of M that has three distinct boundary points with respect to M, these boundary points being the points $\bar{A}$, P, and $\bar{B}$.

If g_1 and g_2 are two arcs of G with a common point then the point-set $g_1 + g_2$ is either a simple continuous arc or a simple closed curve. For otherwise one of the arcs g_1 and g_2 (call it g) would contain a point P which is not an endpoint of g but which is a limit point of $M - g$.

Let A and B denote two definite distinct points of M and let AB denote a definite arc of the set G having A and B as endpoints. Let K_C ($C = A, B$) denote the set of all points X of M such that C and X are the extremities of a simple continuous arc which is a subset of M but which contains no point except C in common with AB. An arc from C to X satisfying these conditions

* A set of points is said to be *continuous* if it is closed and connected and contains more than one point.

† The continuous set of points N is said to be a *continuous set of condensation* of the set M if N is a proper subset of M and every point of N is a limit point of $N - M$.

‡ Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 174–176.

§ Ibid., vol. 23 (1917), pp. 233–236.

will be called the arc CX. If neither K_A nor K_B exists then M is the arc AB. Suppose that K_A exists. If X_1 and X_2 are points of K_A, X_1 is said to precede X_2 if AX_1 is a subset of AX_2. It is clear that of any two distinct points of K_A one of them precedes the other one. The set $K_A + A + B$ is closed. For suppose there exist two points O and $\overline{O}$, distinct from A, which are limit points of K_A but which do not belong to K_A. It can easily be proved with the help of Theorem 5 of F.A. that there exist two sets of points $A_1, A_2, A_3, \cdots$ and $\overline{A}_1, \overline{A}_2, \overline{A}_3, \cdots$ belonging to K_A such that, for every n, A_n precedes $\overline{A}_n$ and $\overline{A}_n$ precedes A_{n+1} and such that O is the sequential limit point of $A_1, A_2, A_3, \cdots$ while $\overline{O}$ is the sequential limit point of $\overline{A}_1, \overline{A}_2, \overline{A}_3, \cdots$. It follows, by an argument similar to that used in establishing the existence of the set t in the proof of Theorem 1, that there exists a continuous set of points $\overline{K}$ containing O such that if Y is a point of $\overline{K}$ then every region about Y contains points of infinitely many of the arcs $A_1 A_2, A_2 A_3$*, $\cdots$. For no value of n does the arc $A_n A_{n+1}$ contain a limit point of the point-set composed of the arcs $A_{n+2} A_{n+3}, A_{n+3} A_{n+4}, \cdots$. Hence $\overline{K}$ contains no point of any of the arcs $A_1 A_2, A_2 A_3, \cdots$. It follows that $\overline{K}$ is a continuous set of condensation of M. But it was shown above that M contains no continuous set of condensation. Thus the supposition that there exist two limit points of $K_A + A$ that do not belong to $K_A + A$ has led to a contradiction. It follows that either $K_A + A + B$ is closed or there exists one and only one point P, distinct from B, which does not belong to K_A but is a limit point of K_A. In the latter case P is a sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ belonging to K_A such that, for every n, P_n precedes P_{n+1} and the point-set $P + A_1 P + P_1 P_2 + P_2 P_3$† $+ \cdots$ is a simple continuous arc from A to P. It follows that P either coincides with B or belongs to K_A. But this is contrary to supposition. Hence the set $K_A + A + B$ is closed.

There are several cases to be considered.

Case I. Suppose that $K_A + A$ is unbounded and that K_B does not exist. The set $K_A + A$ contains a countably infinite set of distinct points $A_1, A_2, A_3, \cdots$ such that $A_1 + A_2 + A_3 + \cdots$ has no limit point. Let $\overline{A}_2$ be the first point of this sequence that follows A_1, $\overline{A}_3$ the first one that follows $\overline{A}_2$, etc. There results an infinite sequence $\overline{A}_1, \overline{A}_2, \overline{A}_3, \cdots$ of points belonging to the set $A_1, A_2, A_3, \cdots$ such that, for every n, $\overline{A}_n$ precedes $\overline{A}_{n+1}$. For each n there exists a region R_{n+1} containing $\overline{A}_{n+1}$ but containing no point of M that does not belong to the interval $\overline{A}_n \overline{A}_{n+2}$ of M. It can be shown that there exists a countable set of distinct points $\overline{B}_1, \overline{B}_2, \overline{B}_3, \cdots$ (where $\overline{B}_1 = B$), with no limit point, and a set of arcs $\overline{B}_1 \overline{B}_2, \overline{B}_2 \overline{B}_3, \overline{B}_3 \overline{B}_4, \cdots$ such that

* Here, for every n, $A_n A_{n+1}$ denotes the interval of the arc AA_n whose endpoints are A_n and A_{n+1}.

† Here, for every n, $P_n P_{n+1}$ denotes the interval of AP_{n+1} whose endpoints are P_n and P_{n+1}.

(1) for every n, $\overline{B}_{n+1}$ is in R_{n+1}, (2) the set $M + \overline{B}_1\overline{B}_2 + \overline{B}_2\overline{B}_3 + \cdots$ is an open continuous curve. The set M is a ray of this curve.

Case II. Suppose that K_A is unbounded and that K_B exists but is bounded. Then K_A and K_B can have no point in common and the set K_B cannot contain more than one last point. If P is a point of K_B which is not a last point then $K_B + B - P$ is not connected. Thus the set $K_B + B$ is a simple continuous arc (see Definition 2). That $(K_B + B) + K_A$ is a ray follows by an argument similar to that employed in Case I.

Case III. Suppose K_A and K_B are both unbounded. In this case the closed sets K_A and K_B have no point in common and the set $K_B + \overline{AB} + K_A$ is evidently an open curve.

Case IV. Suppose K_A is bounded and that K_B either is bounded or does not exist and that K_A and K_B have no point in common. In this case M is clearly a simple continuous arc.

Case V. Suppose that K_A is bounded and that K_B and K_A have at least one point in common. It is clear that in this case $K_A + B$ is a simple continuous arc from A to B and that M is a simple closed curve.

If the term simple continuous curve is applied only to point-sets which are either arcs, closed curves, open curves, or rays then it is clear in view of the above results that these point-sets may be defined as follows.

DEFINITION 5. A *simple continuous curve** is a continuous point-set M no continuous subset of which has more than two boundary points with respect to M.

DEFINITION 6. A *simple closed curve* is a continuous point-set M every proper continuous subset of which has just two boundary points with respect to M.

DEFINITION 7. A *simple continuous open curve* is a continuous point-set M every bounded continuous subset of which has just two boundary points with respect to M.

DEFINITION 8. A *simple continuous arc* is a continuous curve M containing a point A such that every continuous proper subset of M that contains A has just one boundary point with respect to M.

DEFINITION 9. A *ray* is a continuous curve M containing a point A such that every bounded continuous subset of M that contains A has just one boundary point with respect to M.

* The term continuous curve is ordinarily not applied to sets that are not bounded. Thus, according to the ordinary terminology, a straight line is not a continuous curve. I suggest that the term continuous curve be applied to every closed point-set which is connected "in kleinem" (in the sense of H. Hahn) whether it be bounded or unbounded. If this terminology is adopted what is now ordinarily called a continuous curve may be characterized as a *bounded continuous curve.*

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